Postcode en Woonplaats:

Jaar van eerste inschrijving:

Tentamen: Numerieke wiskunde 2

Datum: 6 - 11 - 2006

Naam docent:

1 a pde: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with $0 < x < 1$, ~~and~~ $t \geq 0$

$u(0,t) = u(m,t) = 0$

and $u(x,0) = \sin(\pi x)$

Taylor:

$u(x_{i+1}, t) = u(x_i + h, t) = u(x_i, t) + h u_x(x_i, t) + \frac{h^2}{2} u_{xx}(x_i, t) + \frac{h^3}{3!} u_{xxx}(x_i, t) + \frac{h^4}{4!} u_{xxxx}(\xi_i^+, t)$

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$u(x_{i-1}, t) = u(x_i - h, t) = u(x_i, t) - h u_x(x_i, t) + \frac{h^2}{2} u_{xx}(x_i, t) - \frac{h^3}{3!} u_{xxx}(x_i, t) + \frac{h^4}{4!} u_{xxxx}(\xi_i^-, t)$

for some ξ_i^+ in $[x_i, x_{i+1}]$ and ξ_i^- in $[x_{i-1}, x_i]$

$u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)$

$=$

$h^2 u_{xx}(x_i, t) + \frac{h^4}{12} u_{xxxx}(\eta_i, t)$ for some $\eta_i \in [x_{i-1}, x_{i+1}]$

$u_{xx}(x_i, t) = \frac{u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)}{h^2} \underbrace{- \frac{h^2}{12} u_{xxxx}(\eta_i, t)}_{\text{local truncation error}}$

$\rightarrow$ order of accuracy is $O(h^2)$ geldt voor ODE nu PDV ?

b $J = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$

the eigenvalues are centered around $-\frac{2}{h^2}$

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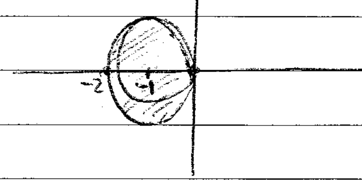
the eigenvalues are in $R = \{z \in \mathbb{C} \mid |z + \frac{2}{h^2}| \leq \frac{2}{h^2}\}$

and because J is symmetric and real, the eigenvalues are in $[-\frac{4}{h^2}, 0]$

c Euler: $w_{i+1} = w_i + h f(t_i, w_i) \rightarrow$ with test equation: $w_{i+1} = w_i + h\lambda w_i = (1 + h\lambda) w_i$

$R = \{h\lambda \in \mathbb{C} \mid |1 + h\lambda| < 1\}$ (region of abs. stability)

(c)



Backward Euler: $w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$

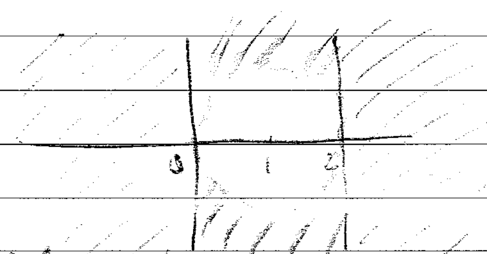
$$w_{i+1} = w_i + h\lambda w_{i+1}$$
$$w_{i+1}(1-h\lambda) = w_i$$
$$w_{i+1} = \frac{1}{1-h\lambda} w_i$$

$$R = \left\{ h\lambda \in \mathbb{C} \;\middle|\; \underbrace{\left|\frac{1}{1-h\lambda}\right| < 1} \right\}$$

$$1 = |1| < |1-h\lambda|$$

for $h\lambda \in [0,2]$, we have $|1-h\lambda| \le 1$

$|1-h\lambda| > 1$:



$|1-z| = 1$

$1-z = e^{i\varphi}$

$z = 1-e^{i\varphi}$ ← cirkel niet stabiel

Trapezoidal method: $w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_{i+1})\right)$

$$w_{i+1} = w_i + \frac{h}{2}(\lambda w_i + \lambda w_{i+1})$$

$$w_{i+1}\left(1 - \frac{h\lambda}{2}\right) = w_i\left(1 + \frac{h\lambda}{2}\right)$$

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$$w_{i+1} = \frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}} w_i$$

$$R = \left\{ h\lambda \in \mathbb{C} \;\middle|\; \underbrace{\left|\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right| < 1} \right\}$$

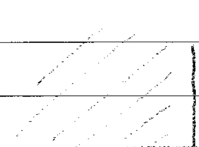
$$\left|1 + \frac{h\lambda}{2}\right| < \left|1 - \frac{h\lambda}{2}\right|$$

$$\cancel{2 + h\lambda} \quad |2+h\lambda| < |2-h\lambda|$$

$h\lambda = 0 \to |2+h\lambda| = |2-h\lambda|$

$h\lambda > 0 \to |2+h\lambda| > |2-h\lambda|$

$h\lambda < 0 \to |2+h\lambda| < |2-h\lambda|$



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1 d ~~the eigenvalues~~ $h\lambda$ must be in $(-2, 0)$

$$\lambda \in \left[-\frac{8k}{h^2}, 0\right] \qquad h = \frac{1}{10} = 0{,}1$$

1

$$-\frac{8k}{h^2} > -2 \rightarrow \frac{8k}{h} < 2$$

$$8k < 20$$

e no, backward Euler and Trapezoidal method are A-stable (see c)

10

2 a A square strictly diagonally dominant matrix → Gaussian elimination possible (without row interchanges)

9

for all multiplier m_{ij} ~~absolute~~ : $|m_{ij}| < 1$

→ unique solution x → A nonsingular

maar onduidelijk zie boek

b first ~~a~~ A is split up in $A = D - L - U$, where D is ~~str~~ a diagonal matrix, $-L$ is a strictly lower triangular matrix and $-U$ is a strictly upper triangular matrix

$$Ax = b \rightarrow (D - L - U)x = b$$

10

$$Dx = (L + U)x + b$$

$$x = D^{-1}(L + U)x + D^{-1}b$$

we start with an initial approximation $x^{(0)}$ to x and iterate : $x^{(k)} = T_j x^{(k-1)} + c_j$, where $T_j = D^{-1}(L + U)$ and $c_j = D^{-1}b$

2 (b) the iteration is continued until convergence or until the maximum number of iterations is exceeded.

5 c $\rho(A) \le \|A\|$ for any natural norm $\|\cdot\|$ [illegible]
$\Rightarrow \rho(A) \le \|A\|_\infty$

$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty \ge \max_i \max_{\|x\|_\infty = 1} \underbrace{\|\lambda_i x\|_\infty}_{= |\lambda_i| \|x\|_\infty}$

$= \max_i |\lambda_i| = \rho(A)$

0 d Jacobi method converges if $\rho(T_j) < 1$

$\rho(T_j) \le \|A\|_\infty < 1 \rightarrow$ convergence

[illegible]

e $\left.\begin{array}{l} x^{(k)} = T_j x^{(k-1)} + c_j \\ x = T_j x + c_j \end{array}\right\} \quad x - x^{(k)} = T_j (x - x^{(k-1)})$

and $x^{(k)} - x^{(k-1)} = T_j (x^{(k-1)} - x^{(k-2)})$

1

$\|x - x^{(k)}\| \le \|T_j\| \|x - x^{(k-1)}\|$

$\|x^{(k)} - x^{(k-1)}\| \le \|T_j\| \|x^{(k-1)} - x^{(k-2)}\|$

$\dfrac{\|x^{(k)} - x^{(k-1)}\|}{\|x^{(k-1)} - x^{(k-2)}\|} \le \|T_j\|$

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3 a power method: start with initial vector x with $\|x\|_\infty = 1$

iterate: $y = Ax$

$\mu = y_p$, where $|y_p| = \|y\|_\infty$

$x = y/y_p$

9

continue until $\|x - y/y_p\|$ is small enough
or until maximum number of iterations is exceeded.

μ gives approximation to λ_1, x gives approximation to $v^{(1)}$ (eigenvector associated with λ_1)

b

1

$\frac{\lambda_2}{\lambda_1} < 1 \rightarrow \left(\frac{\lambda_2}{\lambda_1}\right)^2 < \frac{\lambda_2}{\lambda_1} \rightarrow$ faster convergence

2 x 20 mul

c $w = \frac{1}{\lambda_1 v_i^{(1)}} \cancel{(a_{i1}, \ldots, a_{in})^t}$

Householder transformation can be used to bring
2 symmetric matrices to tridiagonal form,
such that it can be used to compute the eigenvalues
with the QR-algorithm.

a 1. $A - sI = QR$, where Q is orthogonal and
R is an upper triangular matrix

2. $A = RQ + sI$

7 this is iterated until convergence

hoe?

s is usually chosen close to an eigenvalue of A in the

3 (a) to converge when λ_2 is close to λ_1 $\left(\frac{\lambda_2}{\lambda_1} \approx 1\right)$

in the standard QR method, $s=0$

4 a newton: $x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$

just right of the maximum near 2.5 $f'(x)$ is negative and very close to 0

$\rightarrow -\frac{f(x)}{f'(x)}$ is very large

$\rightarrow$ the next x lies far from zero 3

b $f(x) = \frac{\sin(\pi x)}{x}$

$$f'(x) = \frac{\pi x \cos(\pi x) - \sin(\pi x)}{x^2}$$

$$g(\lambda, x) = f(x) + (1-\lambda) f(x_0)$$

$$Dg(\lambda, x) = g_\lambda(\lambda, x) + g_x(\lambda, x) \cdot x'(\lambda) = 0$$

$$g_\lambda = -f(x_0)$$

$$g_x = f'(x)$$

$$\rightarrow x'(\lambda) = -\frac{g_\lambda(\lambda, x)}{g_x(\lambda, x)} = \frac{f(x_0)\, x^2}{\pi x \cos(\pi x) - \sin(\pi x)}$$

$\lambda \in [0,1]$, $x(0) = x_0$

c we get an approximation for x